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Quantum mechanical exponential operators' disentangling by virtue of matrices' $L\mathfrak{D}U$ decomposition

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Abstract
By using $L\mathfrak{D}U$ decomposition of the 3×3 real matrix and Dirac's coordinate representation we can derive disentangling of some exponential operators, which is useful for analyzing both squeezing and entangling mechanisms. This method may be generalized to the $n \times n$ real matrix case.

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1. Introduction

In [1], Dirac indicated that to correspond to classical canonical transformations regarding coordinate momentum in phase space there usually exist quantum mechanical unitary operators. In [2], it is pointed out that the technique of integration within an ordered product (IWOP) of operators [3, 4] may establish a direct 'bridge' connecting these two kinds of transformations. For example, in [2] we have shown that a classical coordinate dilation $q_1 \to q_1/\mu$ may engender the unitary single-mode squeezing operator $\mu^{-1/2}\int_{-\infty}^{\infty} \mathrm{d}q_1 |q_1/\mu\rangle\langle q_1| = \exp\left\{\left(-\frac{a_1^{\dagger 2}}{2}\tanh\lambda\right)\exp\left[\left(a_1^\dagger a_1 + \frac{1}{2}\right)\ln \operatorname{sech}\lambda\right]\exp\left\{\left(\frac{a_1^2}{2}\tanh\lambda\right)\right.\right.$, where $|q_1\rangle$ is the eigenvector of coordinate operator Q_1, $Q_1|q_1\rangle = q_1|q_1\rangle$, $Q_1 = \left(a_1 + a_1^\dagger\right)/\sqrt{2}$, $\mu = \mathrm{e}^\lambda$. In this work we shall extend this idea (method) to a more complicated case. We shall examine the quantum mechanical operator corresponding to the classical transformation $q_i \to \Lambda_{ij} q_j$ in the 3-mode coordinate representation, where Λ is a 3×3 real matrix. We then use matrices' $L\mathfrak{D}U$ decomposition to derive exponential operators' disentangling. This formalism may be generalized to the $n \times n$ real matrix case. The work is arranged as follows. In section 2, we discuss decomposition of the exponential operator corresponding to the transformation $\binom{q_1}{q_2} \to \begin{pmatrix} A & B \\ C & D\end{pmatrix}\binom{q_1}{q_2}$. In section 3 we extend this to the three-dimensional case. The decomposition is useful for analyzing both the squeezing and entangling mechanisms.

2. Exponential operator disentangling corresponding to $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ transformation

We employ Dirac's coordinate representation to map the classical transform to its quantum image,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \to \sqrt{\det(AD-CB)}\int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right| \equiv \hat{K}, \tag{1}$$

where A, B, C, and D are real parameters, $\sqrt{\det(AD-CB)}$ is anticipated to guarantee $\hat{K}$'s unitarity, in Fock space $|q_i\rangle$ is expressed as

$$|q_i\rangle = \pi^{-1/4}\exp\left\{-\frac{q_i^2}{2} + \sqrt{2}q_i a_i^\dagger - \frac{a_i^{\dagger 2}}{2}\right\}|0\rangle_i\,, \qquad i = 1, 2, \tag{2}$$

where $Q_i\,|q_i\rangle = q_i\,|q_i\rangle$, $Q_i = \left(a_i + a_i^\dagger\right)/\sqrt{2}$, $|0\rangle_i$ is the vacuum state annihilated by the Bose annihilation operator a_i, $\left[a_i, a_i^\dagger\right] = 1$. Once the integration in (1) is performed with the use of the IWOP technique, then the explicit form of unitary operator $\hat{K}$, which corresponds to the classical transform $q_1 \to Aq_1 + Bq_2, q_2 \to Cq_1 + Dq_2$, can be obtained. On the other hand, according to the matrices' $L\mathfrak{D}U$ decomposition theory [5] (where L means a lower triangular unit matrix, $\mathfrak{D}$ is a diagonal matrix, while U is an upper triangular unit matrix),

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{C}{A} & 1 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \frac{AD-CB}{A} \end{pmatrix}\begin{pmatrix} 1 & \frac{B}{A} \\ 0 & 1 \end{pmatrix}, \tag{3}$$

and using the orthogonality $\delta(q_i - q_i') = \langle q_i'|q_i\rangle, \langle q_i|P_i = -\mathrm{i}\frac{\partial}{\partial q_i}\langle q_i|$, we have $\hat{K}$'s disentangling formula

$$\hat{K} \equiv \hat{L}'\hat{\mathfrak{D}}'\hat{U}', \tag{4}$$

where

$$\hat{L}' = \int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left|\begin{pmatrix} 1 & 0 \\ \frac{C}{A} & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right| = \exp\left(-\mathrm{i}\frac{C}{A}Q_1P_2\right), \tag{5}$$

$$\begin{aligned}\hat{\mathfrak{D}}' &= \sqrt{\det(AD-CB)/A}\sqrt{A}\int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left|\begin{pmatrix} A & 0 \\ 0 & \frac{AD-CB}{A} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_1 \end{pmatrix}\right| \\ &= \exp\left[-\mathrm{i}\left(Q_1P_1 + P_1Q_1\right)\ln\sqrt{A} - \mathrm{i}\left(Q_2P_2 + P_2Q_2\right)\ln\sqrt{\frac{AD-CB}{A}}\right]\end{aligned} \tag{6}$$

is a squeezing operator [6], and

$$\hat{U}' = \int_{-\infty}^{\infty} \mathrm{d}q_1\,\mathrm{d}q_2 \left|\begin{pmatrix} 1 & \frac{B}{A} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right| = \exp\left(-\mathrm{i}\frac{B}{A}Q_2P_1\right). \tag{7}$$

Thus matrices' $L\mathfrak{D}U$ decomposition in matrix theory is useful to us to find new operator formulae. Note that one can also decompose

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & \frac{B}{D} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{AD-BC}{D} & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} 1 & 0 \\ C/D & 1 \end{pmatrix}, \tag{8}$$

then the corresponding exponential operator's disentangling takes another form.

3. Exponential operator disentangling corresponding to a 3 × 3 transform matrix

We begin by constructing the following ket–bra in an integration form,

$$\hat{V} \equiv \sqrt{\det v} \int_{-\infty}^{\infty} \mathrm{d}\vec{q} \left| v \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \qquad v \equiv \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}, \tag{9}$$

where all v_{ij} are real. $\sqrt{\det v}$ is anticipated to guarantee $\hat{V}$'s unitarity.

In matrix theory the sufficient and necessary condition for an $n \times n$ square matrix W uniquely decomposed as $W = L\mathfrak{D}U$ is its principle minor sequence $\Delta_k \neq 0\,(k = 1, 2, \ldots, n)$, and $\mathfrak{D} = \mathrm{diag}\,(d_1, d_2, \ldots d_n)$, $d_k = \Delta_k/\Delta_{k-1}$, $k = 1, 2, \ldots, n$, $\Delta_0 = 1$; in this case we see that this type of decomposition for v is

$$\begin{aligned} v &= \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix} \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \equiv l\mathfrak{d}\mathfrak{u}, \\ \det v &= d_{11}d_{22}d_{33}, \end{aligned} \tag{10}$$

where

$$\begin{aligned} d_{11} &= v_{11}, \qquad d_{22} = \frac{v_{22}v_{11} - v_{21}v_{12}}{v_{11}}, \\ d_{33} &= \frac{v_{33}v_{11} - v_{13}v_{31}}{v_{11}} - \frac{v_{23}v_{11} - v_{13}v_{21}}{v_{11}} \frac{v_{32}v_{11} - v_{31}v_{12}}{v_{11}v_{22} - v_{12}v_{21}} \end{aligned} \tag{11}$$

and

$$\begin{aligned} l_{21} &= \frac{v_{21}}{v_{11}}, \qquad l_{31} = \frac{v_{31}}{v_{11}}, \qquad l_{32} = \frac{v_{32}v_{11} - v_{31}v_{12}}{v_{11}v_{22} - v_{12}v_{21}}, \\ u_{12} &= \frac{v_{12}}{v_{11}}, \qquad u_{13} = \frac{v_{13}}{v_{11}}, \qquad u_{23} = \frac{v_{23}v_{11} - v_{13}v_{12}}{v_{11}v_{22} - v_{12}v_{21}}. \end{aligned} \tag{12}$$

Following the orthogonality $\langle \vec{q}|\vec{q}\,'\rangle = \delta(\vec{q} - \vec{q}\,')$, we have

$$\hat{V} = \hat{L}\hat{\mathfrak{D}}\hat{U}, \tag{13}$$

where

$$\hat{L} = \int \mathrm{d}^3\vec{q} \left| l \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \qquad l \equiv \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}, \tag{14}$$

$$\hat{\mathfrak{D}} = \sqrt{\det \mathfrak{d}} \int \mathrm{d}^3\vec{q} \left| \mathfrak{d} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \qquad \mathfrak{d} \equiv \begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & 0 \\ 0 & 0 & d_{33} \end{pmatrix}, \tag{15}$$

$$\hat{U} = \int \mathrm{d}^3\vec{q} \left| u \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \qquad u \equiv \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}. \tag{16}$$

Splitting l and u as

$$l = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & l_{32} & 1 \end{pmatrix} \equiv fh, \tag{17}$$

$$u = \begin{pmatrix} 1 & 0 & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv ys, \tag{18}$$

respectively, we see

$$\hat{L} \equiv \hat{F}\hat{H}, \qquad \hat{U} \equiv \hat{Y}\hat{S}. \tag{19}$$

Here

$$\hat{F} = \int \mathrm{d}^3\vec{q} \left| f \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \tag{20}$$

$$\hat{H} = \int \mathrm{d}^3\vec{q} \left| h \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \tag{21}$$

and

$$\hat{Y} = \int \mathrm{d}^3\vec{q} \left| y \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|, \tag{22}$$

$$\hat{S} = \int \mathrm{d}^3\vec{q} \left| s \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right|. \tag{23}$$

Therefore, combining (13)–(23) together we obtain

$$\hat{V} \equiv \hat{F}\hat{H}\hat{D}\hat{Y}\hat{S}. \tag{24}$$

In order to derive the explicit forms of $\hat{F}$, $\hat{H}$, $\hat{D}$, $\hat{Y}$ and $\hat{S}$, using the relation

$$\langle \vec{q}|\det G \int \mathrm{d}^3\vec{q}\,' \left|G\vec{q}\,'\right\rangle\langle\vec{q}\,'|\,\varphi\rangle = \int \mathrm{d}^3\vec{q}\,'\delta(\vec{q}\,' - G^{-1}\vec{q})\varphi(\vec{q}\,') = \varphi(G^{-1}\vec{q}), \tag{25}$$

where G is a 3×3 real matrix, and noting

$$f^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{pmatrix}, \qquad h^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{pmatrix}, \tag{26}$$

we see

$$\begin{aligned} &\left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right| \int \mathrm{d}^3\vec{q}\,' \left| f \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right| \varphi \right\rangle \\ &\quad = \varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ -l_{31} & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right) = \varphi(q_1, q_2 - l_{21}q_1, q_3 - l_{31}q_1) \\ &\quad = \mathrm{e}^{-(l_{21}q_1\frac{\partial}{\partial q_2} + l_{31}q_1\frac{\partial}{\partial q_3})}\varphi(q_1, q_2, q_3), \end{aligned} \tag{27}$$

so

$$\hat{F} \equiv \mathrm{e}^{-\mathrm{i}(l_{21}Q_1P_2 + l_{31}Q_1P_3)}. \tag{28}$$

Similarly, we have

$$\left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right| \int \mathrm{d}^3\vec{q}\,' \left| h \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right| \varphi \Big\rangle$$

$$= \varphi\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right) = \varphi(q_1, q_2, q_3 - l_{32} q_2)$$

$$= \mathrm{e}^{-l_{32} q_2 \frac{\partial}{\partial q_3}} \varphi(q_1, q_2, q_3), \tag{29}$$

so

$$\hat{H} \equiv \mathrm{e}^{-\mathrm{i} l_{32} Q_2 P_3}. \tag{30}$$

Further, from

$$y^{-1} = \begin{pmatrix} 1 & 0 & -u_{13} \\ 0 & 1 & -u_{23} \\ 0 & 0 & 1 \end{pmatrix}, \qquad s^{-1} = \begin{pmatrix} 1 & -u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

and

$$\left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right| \int \mathrm{d}^3\vec{q}\,' \left| y \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right| \varphi \Big\rangle$$

$$= \varphi\left(\begin{pmatrix} 1 & 0 & -u_{13} \\ 0 & 1 & -u_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right) = \varphi(q_1 - u_{13} q_3, q_2 - u_{23} q_3, q_3)$$

$$= \mathrm{e}^{-(u_{13} q_3 \frac{\partial}{\partial q_1} + u_{23} q_3 \frac{\partial}{\partial q_2})} \varphi(q_1, q_2, q_3), \tag{32}$$

we derive

$$\hat{Y} \equiv \mathrm{e}^{-\mathrm{i}(u_{13} Q_3 P_1 + u_{23} Q_3 P_2)}; \tag{33}$$

while from

$$\left\langle \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \right| \int \mathrm{d}^3\vec{q}\,' \left| s \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right\rangle \left\langle \begin{pmatrix} q_1' \\ q_2' \\ q_3' \end{pmatrix} \right| \varphi \Big\rangle$$

$$= \varphi\left(\begin{pmatrix} 1 & -u_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}\right) = \varphi(q_1 - u_{12} q_2, q_2, q_3)$$

$$= \mathrm{e}^{-u_{12} q_2 \frac{\partial}{\partial q_1}} \varphi(q_1, q_2, q_3) \tag{34}$$

we are led to

$$\hat{S} \equiv \mathrm{e}^{-\mathrm{i} u_{12} Q_2 P_1}. \tag{35}$$

Now we turn to equation (15), from (2)–(3) and [2, 6] we know that $\hat{\mathfrak{D}}$ is a squeezing operator,

$$\hat{\mathfrak{D}} = \exp\left[-\mathrm{i} \sum_{i=1}^{3} (Q_i P_i + P_i Q_i) \ln \sqrt{d_{ii}}\right]. \tag{36}$$

Combining (28), (30), (32), (35) and (36) together, we have

$$\begin{aligned}\hat{V} &= \hat{F}\hat{H}\hat{D}\hat{Y}\hat{S} \\ &= \mathrm{e}^{-\mathrm{i}Q_1(l_{21}P_2+l_{31}P_3)}\,\mathrm{e}^{-\mathrm{i}l_{32}Q_2P_3}\exp\left[-\mathrm{i}\sum_{i=1}^{3}(Q_iP_i+P_iQ_i)\ln\sqrt{d_{ii}}\right] \\ &\quad\times \mathrm{e}^{-\mathrm{i}Q_3(u_{13}P_1+u_{23}P_2)}\,\mathrm{e}^{-\mathrm{i}u_{12}Q_2P_1},\end{aligned} \tag{37}$$

this is the disentangling of $\hat{V}$.

The above analysis helps to reveal both entangling and squeezing mechanism explicitly. Taking (5) for example, we want to show how $\exp\left(-\mathrm{i}\frac{C}{A}Q_1P_2\right)$ plays the role of both entangling and squeezing when it operates on the direct-product state $|p\rangle_1\otimes|q\rangle_2$, where $|p\rangle_1$ is the momentum eigenstate of the first mode. Clearly, $\exp\left(-\mathrm{i}\frac{C}{A}Q_1P_2\right)$ is equivalent to the result that $\mathrm{e}^{-\mathrm{i}Q_1P_2}$ undergoes a single-mode squeezing transform, i.e.,

$$\mathrm{e}^{-\mathrm{i}\frac{C}{A}Q_1P_2} = \mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}\,\mathrm{e}^{-\mathrm{i}Q_1P_2}\,\mathrm{e}^{-\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}, \tag{38}$$

and $\mathrm{e}^{-\mathrm{i}Q_1P_2}$ is an entangling operator which entangles $|p\rangle_1\otimes|q\rangle_2$ to become the bipartite entangled state $|\eta\rangle$ [7]

$$\mathrm{e}^{-\mathrm{i}Q_1P_2}|p\rangle_1\otimes|q\rangle_2 = \left[\frac{\mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}}{\sqrt{2\pi}}|\eta\rangle\right]_{\eta_1=-q,\,\eta_2=p}, \tag{39}$$

where

$$|\eta\rangle = \exp\left[-\tfrac{1}{2}|\eta|^2+\eta a_1^\dagger-\eta^*a_2^\dagger+a_1^\dagger a_2^\dagger\right]|00\rangle \tag{40}$$

is the common egenvector of two particles' relative position and the total momentum,

$$(Q_1-Q_2)\,|\eta\rangle=\eta_1|\eta\rangle,\qquad (P_1+P_2)|\eta\rangle=\eta_2|\eta\rangle,\qquad \eta=\tfrac{1}{\sqrt{2}}\,(\eta_1+i\eta_2)\,. \tag{41}$$

Experimentally, an ideal $|\eta\rangle$ state can be generated by a 50/50 beamsplitter when two input light beams are maximally squeezed in Q-direction and P-direction, respectively. To see the correctness of (39) we use the Schmidt decomposition of $|\eta\rangle$ [8],

$$|\eta\rangle = \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle_1\otimes|q-\eta_1\rangle_2\,\mathrm{e}^{\mathrm{i}q\eta_2}, \tag{42}$$

it follows

$$\begin{aligned}\mathrm{e}^{\mathrm{i}Q_1P_2}|\eta\rangle &= \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle_1\otimes\mathrm{e}^{\mathrm{i}qP_2}\,|q-\eta_1\rangle_2\,\mathrm{e}^{\mathrm{i}q\eta_2} \\ &= \mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\int_{-\infty}^{\infty}\mathrm{d}q\,|q\rangle_1\mathrm{e}^{\mathrm{i}q\eta_2}\otimes|-\eta_1\rangle_2 \\ &= \sqrt{2\pi}\,\mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}\,|p=\eta_2\rangle_1\otimes|-\eta_1\rangle_2\,.\end{aligned} \tag{43}$$

Operating $\mathrm{e}^{-\mathrm{i}Q_1P_2}$ on both sides of (43) results in (39). Noting

$$\mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}|q\rangle_1=\sqrt{\frac{A}{C}}\left|\frac{A}{C}q\right\rangle_1,\qquad \mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}|p\rangle_1=\sqrt{\frac{C}{A}}\left|\frac{C}{A}p\right\rangle_1, \tag{44}$$

so using (38), (44), (39) and (42), one by one, we can see how $\exp\left(-\mathrm{i}\frac{C}{A}Q_1P_2\right)$ plays the role of both entangling and squeezing, i.e.,

$$\begin{aligned}\exp\left(-\mathrm{i}\frac{C}{A}Q_1P_2\right)|p\rangle_1\otimes|q\rangle_2 &= \mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}\,\mathrm{e}^{-\mathrm{i}Q_1P_2}\sqrt{\frac{A}{C}}\left|\frac{A}{C}p\right\rangle_1\otimes|q\rangle_2 \\ &= \sqrt{\frac{A}{C}}\,\mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}}\left[\frac{\mathrm{e}^{-\mathrm{i}\eta_1\eta_2/2}}{\sqrt{2\pi}}|\eta\rangle\right]_{\eta_1=-q,\,\eta_2=\frac{A}{C}p}\end{aligned}$$

$$= \sqrt{\frac{A}{C}}\, \mathrm{e}^{\mathrm{i}(Q_1P_1+P_1Q_1)\ln\sqrt{C/A}} \left[\frac{\mathrm{e}^{-\mathrm{i}\eta_1\eta_2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}q'|q'\rangle_1 \otimes |q'-\eta_1\rangle_2\, \mathrm{e}^{\mathrm{i}q'\eta_2}\right]_{\eta_1=-q,\,\eta_2=\frac{A}{C}p}$$
$$= \left[\frac{\mathrm{e}^{-\mathrm{i}\eta_1\eta_2}}{\sqrt{2\pi}} \frac{A}{C} \int_{-\infty}^{\infty} \mathrm{d}q' \left|\frac{A}{C}q'\right\rangle_1 \otimes |q'-\eta_1\rangle_2\, \mathrm{e}^{\mathrm{i}q'\eta_2}\right]_{\eta_1=-q,\,\eta_2=\frac{A}{C}p}$$
$$= \frac{\mathrm{e}^{\mathrm{i}q\frac{A}{C}p}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}q''|q''\rangle_1 \otimes \left|\frac{C}{A}q''+q\right\rangle_2 \mathrm{e}^{\mathrm{i}q''p}, \tag{45}$$

which is a more complicated entangled state; when $\frac{C}{A} = 1$, (45) reduces to (39) as expected.

In conclusion, the $L\mathfrak{D}U$ decomposition of matrices can be mapped onto operators' product in Hilbert space by using appropriate quantum mechanical representations, so their quantum images are also decomposable. The decomposition is useful in analyzing both squeezing and entangling mechanisms. This method combining with the IWOP technique can lead people to derive many new operator identities and can be generalized to the case of $n \times n$ matrices.

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